64. Reflections on mathematical concepts as starting points for mathematical thinking

In: R. Biehler, R. W. Scholz, R. Strässer, B. Winkelmann (Eds.), Didactics of Mathematics as a Scientific Discipline, Dordrecht, Boston, London, 1993, 61-72.

1. Introduction

1.1 Mathematics didactics in teacher education for Gymnasium

Students preparing to teach mathematics at the Gymnasium (see WEIDIG 1992) in Germany traditionally have to master a complete university education in mathematics. This means that they are introduced to calculus, linear algebra, analytical geometry, theory of functions, algebra, number theory, differential geometry, differential equations, probability and statistics, numerical mathematics, etc. This mathematics is far beyond the elementary mathematics they will have to teach as future teachers. But the idea of this type of education is that teachers can only present elementary mathematics at the Gymnasium in a valid manner, if they are familiar with the higher mathematics behind it. "Elementary Mathematics from an Advanced Standpoint" by F. KLEIN (1968) made this notion explicit: a mathematics education of this type should make the future teachers think *mathematically*.

But F. KLEIN also saw the need for lectures about the didactics of mathematics in teacher education to help student teachers to think *didactically*. This was supported by other university mathematicians such as A. PRINGSHEIM. As a result lectures in didactics of mathematics were offered at some universities (GRIESEL , STEINER 1992). This development was continued in the sixties by mathematicians such as H. BEHNKE, H. KUNLE, D. LAUGWITZ and G. PICKERT, who invited experienced teachers to offer lectures in didactics of mathematics. It turned out that these lectures stimulated research in didactics of mathematics,

and that the growing didactical research helped to improve these lectures. Very typical were H.-G. STEINER's lectures in Münster. His lecture on the foundations of geometry from a didactical point of view was published in 1966 (STEINER 1966a). During the following decades didactical theories for most of the mathematical subject areas of the Gymnasium in Germany were developed, e.g. algebra (VOLLRATH 1974), calculus linear algebra and stochastics (TIETZE, KLIKA , WOLPERS 1982), calculus (BLUM , TÖRNER 1983), numerical mathematics (BLANKENAGEL 1985), geometry (HOLLAND 1988), stochastics (BOROVCNIK 1992).

1.2 Reflecting on concepts in lectures on didactics of mathematics

In their mathematical education student teachers are expected to acquire hundreds of mathematical concepts, to become acquainted with properties of these concepts through hundreds of theorems, and to solve problems involving these concepts. Relatively few of these concepts are relevant for their future teaching. It turns out that their knowledge of these concepts is often as vague as their knowledge of concepts in general. But for teaching, their metaknowledge about concepts is absolutely insufficient. Lectures on didactics of mathematics therefore have to reflect on concepts as they affect teaching. And this can be a starting point for didactical thinking.

Questions should be discussed with student teachers which can help them to arrive at central problems of didactics of mathematics. This paper reports about questions on *concept teaching and learning*. It will show how students' reflections about their experience with mathematics lead to basic problems of concept learning and teaching, and how elements of a theory of concept teaching can give the student teachers a perspective for their future teaching.

Elements of a theory of concept teaching, as I understand it, were offered in my book "Methodik des Begriffslehrens im Mathematikunterricht" (1984), which was the result of empirical and analytical research on concept teaching. This

research has been continued in recent years. I want to show in this paper how it was stimulated by discussions with student teachers, and vice versa how this research has stimulated the discussions.

Many student teachers contributed to this research by investigations connected with a thesis for their examination. As a side effect, most of my student teachers felt that the lectures in didactics of mathematics also helped them to understand their "higher" mathematics better.

2 Starting points for didactical thinking

2.1 Evaluation of mathematical concepts

At the beginning of my lectures on didactics of calculus I usually ask my student teachers: "What are the central concepts of calculus?" They suggest concepts like real number, function, derivative, integral, limit, sequence, series, etc. At some point a discussion starts whether a certain concept is "central". This can happen with concepts such as boundary, monotony, accumulation point, etc. Ultimately the students feel a need for a discussion about the meaning of the term "central concept". Obviously there is no definition for this term. But one can argue for a certain concept to be central or not. For example calculus is about functions. But calculus deals with functions in a specific manner: one is interested in the derivative and in the integral of functions. Forming these concepts was the beginning of calculus in history. But for a certain class of functions the derivative and the integral can be found algebraically. Calculus really starts at functions which need limits to find the derivatives and the integral. Therefore one could say that the central concept is the concept of limit. (Although calculus without limits is possible to some extent, e.g. LAUGWITZ 1973.) On the other hand, the concept of limit needs the concepts of real number and function, which can therefore also be called "central concepts".

One might think that this is a rather academic discussion. But questions like this are essential when one plans a calculus course for the Gymnasium. A key

problem then is the *choice of concepts* which have to be taught in this course. This calls for an evaluation of concepts in the context of teaching (This might lead to different results!).

There seems to be a tendency to put too much emphasis on the use of a concept. But OTTE has pointed out that concepts have to be seen both as objects and tools. Therefore concepts offer both *knowledge and use*. An adequate evaluation of concepts from the standpoint of teaching therefore has to take into account both these properties and how they complement each other. OTTE and STEINBRING worked this out for the concept of continuity (1977); FISCHER compared the concepts of continuity and derivative from this point of view (1976). One important approach to the evaluation process is through historical analysis of the development of the concept, which incorporates intentions, definitions, properties, applications, etc (see: chapter 8).

For example: Concept formation is very often embedded in problem solving. A historical analysis of the relationship between concept formation and problem solving can reveal different roles which concepts can play (VOLLRATH 1986). Infinite series were introduced as *instruments for solving problems* of calculating areas of surfaces. But infinite series also became *solutions of problems* when they were used to develop functions into series, e.g. sine, logarithms. When the concept of infinite series was established in calculus it turned out to be a *source of new problems*. The critical conceptual work in infinite series became an *aid for precising the problem* of "infinite addition". The concept of absolutely convergent series, with the possibility of rearranging the terms, served as *means for guaranteeing a certain method*.

This analysis shows different possibilities for embedding concept teaching into problem solving processes. Obviously this gives rise to specific conceptual images through the process of teaching. By these considerations the student teachers can get an idea of a genetic problem-oriented approach to the teaching of concepts. The perspective of *different roles of concepts* can help them to build up a repertoire of different modes of concept teaching in mathematics

education.

When a mathematical concept is taught in school the students are not only expected to understand it, but also to know its importance (WINTER 1983). Investigations show (VOLLRATH 1988) that there are different ways for the teacher to express his own appreciation of a concept. Explicit expressions based on reasons seem to be most effective. But the future teachers must learn to accept students' evaluations as expressions of their personality also when they differ from their own appreciation of a concept.

2.2 Relationships between mathematical concepts

During our discussion about the central concepts of calculus we refer to relationships between concepts. This can be the starting point for further investigations (VOLLRATH 1973). For example I ask my student teachers for the different types of sequences. A possible collection is: rational sequence, real sequence, constant sequence, arithmetical sequence, geometrical sequence, convergent sequence, zero-sequence, bounded sequence, increasing sequence, decreasing sequence, finally constant sequence, Cauchy-sequence, convergent sequence with rational limit, etc. We then try to get an overview. Theorems such as:

Every convergent sequence is bounded.

or: Every increasing and bounded sequence is convergent.

lead to a hierarchy of concepts (VOLLRATH 1973). The student teachers discover that knowledge of calculus does not only mean knowledge of concepts but also of relationships between concepts. They become aware of the importance of *networked learning*.

The study of the hierarchy of concepts leads to the didactical problem of *arranging the concepts* for teaching in school. In a first approach different teaching sequences are formed and discussed from the point of view of teaching and learning. But it is also necessary to provide opportunities for the students to

discover relationships between concepts.

From a systematic point of view it seems convenient to start with the most general concept and to arrive at special concepts. But there can also be reasons for going the opposite way. There has been a long discussion in pedagogy whether one should procede from the general to the special or vice versa. Didacticians know that this question is too general. Didactics of mathematics is looking for more specific answers. More particularly, didacticians agree that there are many different ways of learning a network of concepts so that the concepts are understood and mastered, and so that the relationship between them is known and can be used.

2.3 Structural analysis of mathematical concepts

Our discussions about the essentials of calculus lead to the real numbers as the basis of calculus. One can then continue the investigation by asking which property of the real numbers is needed to satisty the specific requirements of calculus. Analysing the central concepts, theorems and proofs of calculus leads to the discovery of the well-known fact, that the real number system is "complete". For most of the students this means that nested intervals always contain one real number. The student teachers will perhaps learn that completeness can also be expressed in terms of Dedekind-sections or Cauchy-sequences. But STEINER (1966 b) showed that completeness has to do not only with the method by which the real numbers are constructed in terms of rational numbers. His paper revealed that completeness is equivalent to the propositions of the fundamental theorems of calculus, e.g. the intermediate value property, the Heine-Borel property or the Bolzano-Weierstrass property. This study helps the student teachers to understand the fundamentals of calculus better.

But the great variety of the 12 different properties expressing completeness in Steiner's paper raises questions relevant to teaching. A starting question could be: Which property should be used in mathematics instruction (grade 9) to

introduce the completeness of the real numbers? And again it is not just the answer which matters, but more importantly the reasoning. Moreover, reasons can refer to both knowledge and use. One can discuss which property offers most knowledge and best use in the easiest way. But although didactics tries to optimize teaching and learning (GRIESEL 1971, p. 73), it must not be neglected that each property reveals a certain aspect of real numbers which emerged at a certain period in the history of the development of the concept.

Although there are different possible approaches, which are equivalent from a systematical point of view, "easy" ways can be misleading. For example defining convexity of a function by its derivatives, or defining logarithm as an integral of 1/x is "putting the cart before the horse" (KIRSCH 1977).

We took this discussion about completeness as an example of a structural analysis which was an interesting didactical problem in the sixties. Things change, nowadays problems of applications of calculus seem to be more interesting. Certainly this change of interest can also be a point of reflection.

2.4 Logical analysis of definitions

When we talk about the definitions of the central concepts of calculus most of my student teachers confess that they have had difficulties in understanding these definitions. We then want to find out the reasons for these difficulties.

Certainly one problem is the complex logical structure of the definitions. Take for example continuity:

A function f is said to be continuous at x_o iff

for all positive ϵ there exists a positive δ, such that,

for all x if $|x - x_o| < \delta$ then $|f(x) - f(x_o)| < \epsilon$.

It is especially the "tower of quantifiers" "for all"..."there exists"... "for all" - and the implication "if...then", which causes the difficulties.

Therefore one would look for equivalent but less complex definitions. Different calculus books help my students to find a lot of definitions and to compare them from the perspective of logical structure. Obviously the difficulties are only shifted by the "simpler" definition:

A function f is said to be continuous in x_o iff

$$\lim_{x \to x_o} f(x) = f(x_o).$$

Now the problems are contained in the definition of the limit.

Discussions like these have a long tradition in the didactics of calculus. There are some psychological findings (e.q. disjunctive definitions are more difficult to learn than conjunctive definition; see CLARK 1971) which can support judgements. But they are not very surprising.

Another possibility is to restrict the concepts of calculus. A very interesting approach is the Lipschitz-calculus (KARCHER 1973), where for example the definition of L-continuity is logically simpler then the definition of continuity in general.

But finally, the whole problem of generalization and formalization in calculus teaching has become problematic. Historical considerations make clear that the epsilon-delta form of the definition is the result of a long process of rigorising which was completed by the end of the last century (FISCHER 1978). Teaching should give students a chance to experience a similar process in concept learning. For this reason there is a renewed interest in more intuitive approaches to calculus in the Gymnasium (e.g. BLUM , KIRSCH 1979). A historical discussion about the development of rigour in calculus can help students to understand better the use of all the "epsilon-delta-stuff" of calculus.

As an excellent example of a stepwise, increasingly precise approach to the concepts of calculus I present to my student teachers the introduction to continuity by OSTROWSKI (1952) where a sequence of trial, critique, further trial,...

finally leads to the epsilon-delta-definition.

2.5 Understanding of concepts

Didactical discussions about concepts soon arrive at the problem of understanding. What does it mean to understand a concept? The first answer of student teachers is usually, "to know a definition". But this answer can easily provoke a discussion. A definition can be learnt by heart without beeing understood. They soon find out that one has to describe understanding of a concept by means of abilities; e.g. to be able to give examples – to give counterexamples – to test examples – to know properties – to know relationships between concepts - to apply knowledge about the concept. Abilities like these can be tested. But it is more difficult to describe what we mean by "having images of a concept", "to appreciate a concept", "knowing the importance of a concept".

Discussions soon lead to the insight that there are *stages of understanding*. This view has a long tradition. And there are also "master pieces" on presenting concepts in stages. A good example is MANGOLDT , KNOPP's introduction to integration. It starts with an intuitive approach on the basis of area functions. After this, integrals are calculated. And in a third stage, a lot of conceptual work on defining integrals is done (1965).

Considerations like these help the students to understand stage-models of understanding (see: DYRSZLAG 1972, VOLLRATH 1974, HERSCOVICS , BERGERON 1983).

The need for better understanding leads to the discovery that there is no final understanding. This is a sort of paradox: understanding is both a goal and a process. And there are further paradoxes of understanding (VOLLRATH 1993). They have their origin in the nature of mathematical knowledge (see: JAHNKE 1978, KEITEL,OTTE , SEEGER 1980, STEINBRING 1988).

2.6 Forming mathematical concepts

The strangest question for my student teachers is: "Have you ever formed a new mathematical concept on your own?" They are generally very puzzled by this question. I always get the answer: "No!" And sometimes they ask me: "Should we have done so?"

For most of the student teachers university education in mathematics means receptive learning. They can be creative to some extent in problem solving when they find a solution, perhaps on the basis of an original idea. But they will never be asked to form a new concept. Some students have perhaps written poems on their own, they have painted pictures, they have composed melodies, they have made biological, chemical, or physical experiments. But why don't they develop mathematics on their own? We all feel that they will have no real chance of inventing an important piece of mathematics. But isn't this also true for their poetry, their painting, their music, their biology, chemistry, or physics? Perhaps it is "the power of the mathematical giants" that discourages students for making mathematics.

As an example, I try to encourage my student teachers to invent a new type of real sequence just by thinking out a certain property. Maybe one chooses as the property of a sequence (a_n):

$$a_n = 0 \text{ for indefinitely many } n.$$

At first one will think of a suitable name for this type of sequence. Let us call it a "stutter-sequence". Does there exist a stutter-sequence? Is every sequence a stutter sequence? These questions ask for examples and counterexamples. What about the sum or the product of stutter sequences? Are they stutter-sequences too? What is the relationship to other sequences? Answers can be formulated as theorems which form a small piece of theory. These steps are routines. But most of my students are not familiar with these routines. How then will they adequately teach their future students about concept formation? Students in general do not think of mathematics as a subject in which they can be creative. Concept

formation offers the possibility of *creative thinking* in mathematics (VOLLRATH 1987).

2.7 Thinking in concepts

From a formalistic point of view, the names of mathematical concepts are arbitrary. But to some extent the name often expresses an image . "Continuous" is a term which bears intuitions. This is also true for terms like "increasing", "decreasing", "bounded" etc. On the other hand, "derivative" and "integral" give no hints to possible meanings. Most of my student teachers are familiar with the fact that a name does not give sufficient information about a concept. But there is some research suggesting that most students in school refer to the meaning of the concept name and not to a definition. There is also research which indicates that images evoked by the everyday meaning of the name are responsible for misunderstanding the concept (VIET 1978, VOLLRATH 1978).

On one hand, students have to learn that the *meaning of a mathematical concept* has to be defined. On the other hand, it is true that certain images, ideas and intentions lead to definitions which stress certain aspects but disregard others. The concept of sequence can be defined as a function defined on the set of natural numbers. This stresses the image of mapping, whereas the idea of succession is left in the background. The same is true for many of the central concepts of calculus. This was pointed out very clearly by STEINER (1969) in his historical analysis of the function concept, and it was investigated for many of these concepts by FREUDENTHAL in his "Didactical Phenomenology" (1983).

2.8 Personal shaping of mathematical concepts

When a mathematician wants to define a concept then there is not much freedom for him to formulate the defining property. Some authors prefer to use formal language, others try to avoid it as much as possible. A comparison of

text books from the same time shows rather little variety of styles. A comparison between text-books with similar objectives of different times reveals more differences. But again, this is more a congruence of developing standards than the expression of different personalities.

However, during the development of an area of mathematics, concept formation is strongly influenced by the leading mathematician at the time. This has been true for calculus. There are fundamental differences in the ways Leibniz and Newton developed calculus. A historical analysis can still identify their different fundamental ideas in modern calculus. The same is true for the theory of functions of a complex variable. One can still see today the different approaches of Riemann and Weierstrass in a modern presentation of the theory. It is possible to speculate with KLEIN that their different "characters" are responsible for the different ways of building up the theory (1926; p. 246). But it is more helpful to concentrate on the differences in experience, intention, and image as the decisive influences on concept formation.

A lecture on the didactics of calculus should give the student teachers an opportunuty to recognize different sources of central parts of the theory, to get acquainted with the mathematicians who pushed forward the development, and to become aware of their motives and images.

Although mathematics has a universal quality when presented in highly developed theories, one should not forget the fact that there are women and men behind it who have influenced the development.

When mathematicians want to learn a new theory they read or hear definitions and at once use certain routines to understand the new concepts. They are at ease when they find that the new concept fits into their existing network of concepts, when it corresponds with their own images, knowledge and experience. They feel resistant to the new concept when they encounter discrepancies. In any case, learning a new concept involves an active process of concept formation. Very often this is accompanied by feelings of interest or resistance. And this is something that the student teachers will often have experienced in their

own mathematical education at the university.

However many of them have the idea that teaching concepts means to present as much knowledge about the concept as they can in as interesting a manner as possible. This is a point at which student teachers can encounter results of communication analysis (ANDELFINGER 1984, VOIGT 1991), which show that students often resist when they are expected to learn new concepts. As a consequence they often form "personal concepts" which differ from their teacher's concepts. And it is surprising that this may occur even though they can solve a lot of problems about the concept correctly. This should sensitize the student teachers to comments made by the students which they will hear when they observe mathematics instruction in their school practice.

2.9 Strategies of concept teaching

Finally, we arrive at a rather delicate problem. When the student teachers look at their own experience as learners of mathematics they all know that there are teachers, professors and authors who are very effective in teaching concepts, whereas others raise many difficulties for the learners. What is the mystery of successful teaching? Is there an optimal way of teaching concepts?

The preceding discussions will protect the student teachers from giving simple answers. They are aware that learning concepts is rather complex. It is not difficult for them to criticize empirical studies testing the effectiveness of "method A" versus "method B". They can also easily identify the weaknesses of investigations about the effectiveness of artifical methods such as those used in psychological testing (e.g. CLARK 1971). They soon find out that one needs a theory of teaching in the background as a basis for making decisions. A good example of such a theory is *genetic teaching* (e.g. WITTMANN 1981), which can be used to give a sense of direction.

To master the complexity of concept teaching the students find that they need to look at the relevant *variables*.

Teaching mathematical concepts has to take into consideration

(i) the students: their cognitive structures, their intellectual abilities, their attitudes, and their needs;

(ii) the concepts: different types of concept, logical structure of definitions, context, development of concepts;

(iii) the teachers: their personality, their intentions, their background.

Behind each of these variables there is a wide variety of theories (see: VOLLRATH 1984). It is impossible to present these theories to the students. However, they can be sensitized to the problems and can get references to literature for further study. Some of these problems can also be touched on in exercises and at seminars.

These considerations help the student teachers to get a differentiated view of teaching: concept teaching has to be planned with respect to these variables.

A reasonable plan for teaching a concept in a certain teaching situation is called a *strategy*. My practise is to look at strategies for teaching concepts by considering different *ranges* of strategies (VOLLRATH 1984). *Local strategies* refer to the plan of a *teaching unit* which is applicable for *standard concepts* like rational function, bounded function, step-function, etc. *Regional strategies* serve for planning the teaching of *key concepts* in *teaching sequences* such as the concept of limit, derivative, or integral of a function.

Global strategies are needed for *leading concepts* which permeate the whole *curriculum*, for example the concept of function is a candidate for such a leading concept.

The student teachers get the opportunity to study models of these types of strategy from "didactical masterpieces" (see also WITTMANN 1984). And they are invited to develop strategies on their own for some examples of different ranges.

Finally, the student teachers should get some hints how to evaluate certain strategies. The most important goal is that they can reason, without being dogmatic. It would be a disaster if didactics of mathematics as a science were to prop up educational dogma.

References

Andelfinger, B., Arithmetische und algebraische Lerner-Konzepte in der S I, Beiträge zum Mathematikunterricht 1984, (pp. 71-74), Bad Salzdetfurth (Franzbecker) 1984

Blankenagel, J.. Numerische Mathematik im Rahmen der Schulmathematik. Mannheim-Wien-Zürich (Bibliographisches Institut) 1985.

Blum, W., Kirsch, A., Zur Konzeption des Analysisunterrichts in Grundkursen, Der Mathematikunterricht, 25 Heft 3 (1979), 6-24

Blum, W., Törner, G., Didaktik der Analysis, Göttingen (Vandenhoeck , Ruprecht) 1983.

Borovcnik, M., Stochastik im Wechselspiel von Intuitionen und Mathematik. Mannheim-Leipzig-Wien-Zürich (Wissenschaftsverlag) 1992.

Clark, D.C., Teaching concepts in the classroom: A set of teaching prescriptions derived from experimental research. Journal of Educational Psychology, 62 (1971), 253-278.

Dyrszlag, Z., Zum Verständnis mathematischer Begriffe 1,2, Mathematik in der Schule, 10 (1972), 36-44; 105-114.

Fischer, R., Fundamentale Ideen bei den reellen Funktionen, Zentralblatt für Didaktik der Mathematik, 8 (1976), 185-192.

Fischer, R., Die Rolle des Exaktifizierens im Analysisunterricht, Didaktik der Mathematik, 6 (1978), 212-226.

Freudenthal, H., Didactical Phenomenology of Mathematical Structures, Dordrecht-Boston-Lancaster (Reidel) 1983.

Griesel, H., Die mathematische Analyse als Forschungsmittel in der Didaktik der Mathematik, In: Beiträge zum Mathematikunterricht 1971 (pp. 72-81), Hannover (Schroedel) 1971.

Griesel, H., Steiner, H.-G., The Organization of didactics of mathematics as a professional field, Zentralblatt für Didaktik der Mathematik, 24 (1992), 287-295.

Herscovics, N., Bergeron J., Models of understanding, Zentralblatt für Didaktik der Mathematik, 15 (1983), 75-83.

Holland, G., Geometrie in der Sekundarstufe, Mannheim-Wien-Zürich (Wissenschaftsverlag) 1988.

Jahnke, H.N., Zum Verhältnis von Wissensentwicklung und Begründung in der Mathematik - Beweisen als didaktisches Problem. Bielefeld: IDM, Bd.10 1978.

Karcher, H., Analysis auf der Schule, Didaktik der Mathematik, 1 (1973), 46-69.

Keitel, Ch., Otte, M., Seeger, F., Text, Wissen, Tätigkeit, Königstein (Scriptor) 1980.

Kirsch, A., Aspects of Simplification in Mathematics Teaching. In: H. Athen , H. Kunle (Eds.), Proceedings of the Third International Congress of Mathematical Education (pp. 98-120). Karls-

ruhe: Organizing Committee of the 3rd ICME 1977.

Klein, F., Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert, part.I. Berlin (Springer) 1926.

Klein, F., Elementarmathematik vom höheren Standpunkte aus, vol.I-III, (Reprint). Berlin (Springer) 1968.

Laugwitz, D., Ist Differentialrechnung ohne Grenzwertbegriff möglich? Mathematisch Physikalische Semesterberichte, 20 (1973), 191-201.

Mangoldt, H. von Knopp, K., Einführung in die höhere Mathematik, Vol.3, (12th edition), Leipzig (Hirzel) 1965.

Ostrowski, A., Vorlesungen über Differential- und Integralrechnung, 1. Basel (Birkhäuser) 1952.

Otte, M., Steinbring, H., Probleme der Begriffsentwicklung - zum Stetigkeitsbegriff, Didaktik der Mathematik, 5 (1977), 16-25.

Steinbring, H., "Eigentlich ist das nichts Neues für Euch!" - Oder: Läßt sich mathematisches Wissen auf bekannte Fakten zurückführen? Der Mathematikunterricht, 34 Heft 2 (1988), 30-43.

Steiner, H.-G., Vorlesungen über Grundlagen und Aufbau der Geometrie in didaktischer Sicht, Münster (Aschendorff) 1966a.

Steiner, H.-G., Äquivalente Fassungen des Vollständigkeitsaxioms für die Theorie der reellen Zahlen, Mathematisch-Physikalische Semesterberichte, XIII (1966b), 180-201.

Steiner, H.-G., Aus der Geschichte des Funktionsbegriffs, Der Mathematikunterricht, 15 Heft 3 (1969), 13-39.

Tietze, U.-P., Klika, M., Wolpers, H., Didaktik des Mathematikunterrichts in der Sekundarstufe II, Braunschweig (Vieweg) (1982).

Viet, U., Umgangssprache und Fachsprache im Geometrieunterricht des 5. und 6. Schuljahres, In: H. Bauersfeld (ed.) Fallstudien und Analysen zum Mathematikunterricht (pp.13-23). Hannover (Schroedel) 1978.

Voigt, J., Das Thema im Unterrichtsprozeß, Beiträge zum Mathematikunterricht 1991.(pp.469-472), Bad Salzdetfurth (Franzbecker) 1991.

Vollrath, H.-J., Folgenringe, Der Mathematikunterricht, 19 Heft4 (1973), 22-34.

Vollrath, H.-J., Didaktik der Algebra, Stuttgart (Klett) 1974.

Vollrath, H.-J., Lernschwierigkeiten, die sich aus dem umgangssprachlichen Verständnis geometrischer Begriffe ergeben, Schriftenreihe des IDM, 18 (1978), 57-73.

Vollrath, H.-J., Methodik des Begriffslehrens im Mathematikunterricht, Stuttgart (Klett) 1984.

Vollrath, H.-J., Zur Beziehung zwischen Begriff und Problem in der Mathematik, Journal für Mathematikdidaktik, 7 (1986), 243-268.

Vollrath, H.-J., Begriffsbildung als schöpferisches Tun im Mathematikunterricht, Zentralblatt für Didaktik der Mathematik, 19 (1987), 123-127.

Vollrath, H.-J., Mathematik bewerten lernen, In: P. Bender (Ed.), Mathematikdidaktik: Theorie und Praxis, Festschrift für Heinrich Winter (pp. 202-209). Berlin (Cornelsen) 1988.

Vollrath, H.-J., Paradoxien des Verstehens von Mathematik, Journal für Mathematikdidaktik, 14 (1993), 35-58

Weidig, I.. On the school system in Germany and the regulation of mathematics teaching. Zentralblatt für Didaktik der Mathematik, 24 (1992), 214-219.

Winter, H., Über die Entfaltung begrifflichen Denkens im Mathematikunterricht, Journal für Mathematikdidaktik, 4 (1983), 175-204

Wittmann, E., Grundfragen des Mathematikunterrichts, (6th ed.). Braunschweig (Vieweg) 1981.

Wittmann, E., Teaching Units as the Integrating Core of Mathematics Education. Educational Studies in Mathematics, 15 (1984), 25-36.

4 Acknowledgements

The considerations of this paper are strongly influenced by the experience of teaching and research in didactics of mathematics for 25 years which I could gain through the promotion of D. Laugwitz, and through stimulating discussions with H.-G. Steiner. By this paper I want to acknowledge Steiner's influence on my work. I have to thank D. Quadling for shaping my English. And I thank the editors for their invitation to write this article.