58. The development of practical arithmetic in lower secondary schools

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1. Practical arithmetic as a topic of applied mathematics

Practical arithmetic is an important topic of applied mathematics taught in lower secondary schools. This problem area has a long tradition in mathematics instruction. But under general discussions about applications during the last decades it has found much interest among didacticians. Starting with considerations about suitable problems and methods the main interest is nowadays directed to an analysis of the underlying processes of thinking, such as mathematization, modelling, and the relationship between mathematics and the real world in general (see BLUM & NISS 1991). This paper intends to expose the development of this general discussion for the special topic of practical arithmetic. Methods of practical arithmetic are in Germany mainly taught in grade 7 in all types of school, but they are initiated in earlier grades and deepened during the following grades.

Practical arithmetic has to do with problems like these:

5 kg apples cost 16.50 DM.
How much do 3 kg apples cost?

or

5 machines can do a job in 12 hours.
How many hours do 3 machines need for the same job?

During the history of mathematics many different methods have been developed to solve problems like these.

2. Traditional practical arithmetic

The ancient *Rechenmeister* (calculation masters) such as ADAM RIESE (1492-1559) taught schemes for solving the problems without telling their students why they work. Giving a scheme for problems from different areas was an improvement, but using it without understanding led to a mechanical drill in mathematics instruction which continued for centuries.

This limitation was overcome at the beginning of our century in writing sentences for finding the solution by reasoning. The problems were solved in this way:

If 5 kg apples cost 16.50 DM,
then 1 kg apples costs the 5th part,
i.e.16.50 DM : 5 = 3.30 DM.
If 1 kg apples costs 3.30 DM,
then 3 kg apples cost 3 times as much,
i.e. 3.30 DM·3 = 9.90 DM.

Only after having understood this kind of reasoning were the students permitted to write their solutions in a shorter formal way like:

5 kg ≙ 16.50 DM
1 kg ≙ 16.50 DM : 5 = 3.30 DM
3 kg ≙ 3.30 DM·3 = 9.90 DM.

The machine problem was solved similarly:

5 machines ≙ 12 hours
1 machine ≙ 12 hours·5 = 60 hours
3 machines ≙ 60 hours :3 = 20 hours.

To distinguish the two cases the students had to test whether it was a case of "the more...the more" or "the more..., the less...".

The problems were presented progressively in different grades.

In grades 3 and 4 simpler problems like

1 kg pears costs 3 DM.
How much do 4 kg pears cost?

or

2 kg plums cost 6 DM.
How much does 1 kg plums cost?

were used as applications of multiplication or division respectively using "denominate numbers". In grades 5 and 6 the student learnt to work with decimal numbers. Finally in grade 7 the general problem was introduced and the scheme was taught explicitly.

The solving of these problems was understood as training for applying mathematics to real life and also for logical thinking (LIETZMANN 1961, BREIDENBACH 1963).

Questions were discussed such as: Should one really be allowed to use short statements like

"5 kg cost 16.50 DM."

instead of

"5 kg apple cost 16.50 DM."

or formal statements like

"5 kg $\hat{=}$ 16.50 DM"?

These didactical problems were handled rather dogmatically. The teachers were committed to a certain notation by their choice of textbook.

For the *Volksschule* the "rule of three" became an almost universal method. The choice of problems and topics in grades 7-9 was determined by the method. Problems and topics from real life that would not correspond to the method were avoided. For example, students would not have found the follo-

wing problem in their textbook, because the scheme would not work:

A 10 g letter costs 20 Pf postage.
How much is the postage for a 30 g letter?

The Gymnasium placed more emphasis on algebraic methods. Therefore proportional situations were often embedded in missing-value problems, using proportions, e.g.

$$\begin{array}{lcl} 5\text{ kg} & \hat{=} & 16.50\text{ DM} \\ 3\text{ kg} & \hat{=} & x\text{ DM} \\ \hline \end{array}$$

$$\frac{5}{3} = \frac{16.5}{x}$$
$$5x = 3 \cdot 16.5$$
$$x = \frac{3 \cdot 16.5}{5}$$

In many cases the proportion was found schematically without any real understanding.

The differences in methods between *Hauptschule* and *Gymnasium* have their origin not only in the different abilities of their students or the variety of accessible technics but rather in their distinct philosophies of education (SCHUPP 1989).

At the end of the 1960s a new discussion arose under the influence of didacticians who analysed the structure of this type of problem. They felt that the way in which these problems were presented and solved in schools could not give the students a chance for a real understanding, because it was mathematically unfounded. They were convinced that a "didactical analysis of structure" would help to develop an adequate mathematical treatment which would guarantee a real understanding.

3. Structural analysis of practical arithmetic

At first the concept of "denominate numbers" was criticized (KIRSCH 1969). It was pointed out that 2 kg, 5 m, 3.5 l, 1½ h and 2.35 DM should be seen as "magnitudes". They belong to domains of magnitudes such as weights, lengths, volumes, times, and quantities of money, in which an operation + and a relation < are defined (GRIESEL 1969, KIRSCH 1969, STEINER 1969). These domains of magnitudes are ordered semi-groups, in which operators can be defined. Certain domains of magnitudes are isomorphic to $(Q^+,+,<)$. They can therefore be used as models for positive rational numbers. This was a justification for working with magnitudes when teaching fractions (KIRSCH 1970). On the other hand fractions can also be understood as rational operators on magnitudes. This idea had been used by H. WEYL in 1918 for a construction of the positive rational numbers. Braunfeld's (1968) model for fractions of "stretchers and shrinkers" became very popular in Germany. As a result of these discussions magnitudes became the basis of teaching practical arithmetic.

Secondly, as we already remarked the "rule of three" was regarded as an instrument for training logical thinking. KIRSCH (1969) pointed out that this has nothing to do with logic. It is really based on the properties of functions:

> A function f is *proportional* if $f(r\cdot x) = r\cdot f(x)$ for all $r \in Q^+$, and all x; it is *antiproportional* if $f(r\cdot x) = 1/r\cdot f(x)$ for all $r \in Q^+$, and all x (x being an element of a domain of magnitudes).

The expression "antiproportional" was introduced by KIRSCH instead of "inversely proportional".

KIRSCH suggested using tables of functions for solving problems on proportional and antiproportional functions.

For example:

Table 1: weight - cost - function

	weight	cost	
	5 kg	16.50 DM	
:5			:5
	1 kg	3.30 DM	
·3			·3
	3 kg	9.90 DM	

Table 2: number - time - function

	number	time	
	5 machines	12 h	
:5			·5
	1 machine	60 h	
·3			:3
	3 machines	20 h	

Many textbooks adopted this method.

As a result of an effective training with fractions, students were expected to solve such problems in a shorter manner:

Table 3: reduced weight-cost function

	weight	cost	
	5 kg	16.50 DM	
$\cdot \frac{3}{5}$			$\cdot \frac{3}{5}$
	3 kg	9.90 DM	

Table 4: reduced number - time - function

number	time
5 machines	12 h
3 machines	20 h

$\cdot\frac{3}{5}$ $\cdot\frac{5}{3}$

This could be called a *scale factor method*.

Students were to discover that proportional functions have constant quotients, whereas antiproportional functions have constant products:

Table 5: constant quotient

weight	cost	cost:weight
5 kg	16.50 DM	3.30 DM/kg
3 kg	9.90 DM	3.30 DM/kg

Table 6: constant product

number	time	number · time
5	12h	60 h
3	20 h	60 h

This property could also be used for problem solving:

Table 7: equal quotients

weight	cost
5 kg	16.50 DM
3 kg	x DM

$$\frac{x}{3} = \frac{16.5}{5}$$
$$x = \frac{16.5 \cdot 3}{3}$$

Table 8: equal products

number	time
5	12 h
3	x h

$$x \cdot 3 = 12 \cdot 5$$
$$x = \frac{12 \cdot 5}{3}$$

The constant quotient of a proportional function could be interpreted as a *unit rate*. This could lead to a further method for solving proportional problems:

5 kg apples cost 16.50 DM.
The unit price is (16.50 : 5)DM/kg = 3.30 DM/kg
3 kg apples cost 3.30 DM/kg·3 kg= 9.90 DM

The unit rate was especially used to compare two given rate pairs, for example:

A package of 5 kg detergent costs 12.00 DM.
A package of 7.50 kg detergent costs 21.00 DM.
Which offer is better?
The unit price for the first offer is 2.40 DM/kg;
for the second offer it is 2.80 DM/kg.
The first offer is better.

The unit rate easily leads to the equation for the function. Let the unit rate of the proportional weight (x) - cost (y) function be 3.30; then

$$y = 3.30 \cdot x.$$

The apple problem can be solved by substituting 3 for x:

$$y = 3.30 \cdot 3 = 9.90.$$

One can use the graphs of the functions for finding solutions. The graph of a proportional function is a straight line through the origin (Figure 1), the graph of an antiproportional function is a hyperbola (Figure 2):

Figure 1: graph of a proportional function

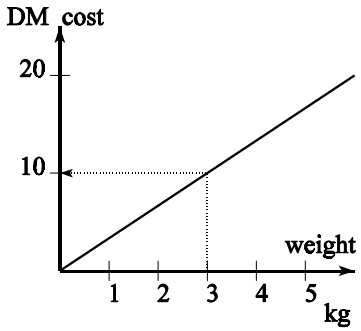
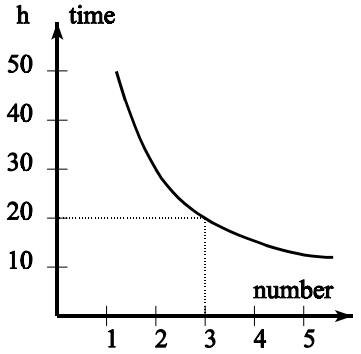


Figure 2: graph of an antiproportional function



Obviously a great variety of methods resulted from regarding the problems from a functional standpoint. The students, especially in the *Gymnasium*, were

expected to learn the different methods and to use them properly in problem situations.

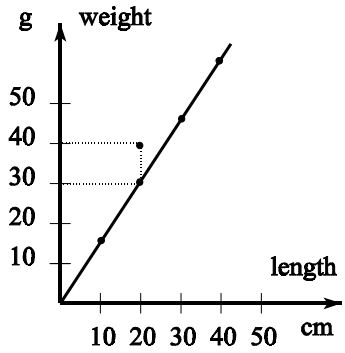
Thirdly, situations were identified which provided new content for mathematics education in which the underlying function is neither proportional nor antiproportional. KIRSCH (1969) made clear that proportionality or antiproportionality is not a property of a situation. One has to test whether it is convenient to describe the situation by a proportional or antiproportional function. If not, one can test different types of functions to find out which fits best. These considerations caused discussions about "modelling" in general (cf. SCHUPP 1989), and stimulated proposals for studying different types of functions. In particular, a great variety of cost functions from commercial life were studied (VOLLRATH 1973). Examples: postage (piecewise constant functions), cost of packages of detergent (discrete functions), cost of fuel including discount for larger quantities (piecewise linear functions). Problems were mainly solved by graphs for these examples. Later on the students also used equations of the functions for the solution of such problems if possible. All these activities made it clear that working with functions is very effective for solving commercial problems.

Studying relations between magnitudes is fundamental in science, especially in physics. Therefore working with functions between domains of magnitudes creates connections between different areas of the school curriculum. The idea of "learning by doing" can be realized in mathematics education through experiments on magnitudes (VOLLRATH 1978). Typical investigations which do not overlap with physics teaching may be: the dependence of the weight of pieces of cardboard on their area, the weight of sets of nails on the number of nails, the weight of a wire on its length, the perimeter of a circle on its diameter. All these activities should include problems which cannot be solved directly but need some consideration of functions. For example, wires of different length and weight but equal material and profile are measured. There is a clamp on one wire. What is its weight? The result of the experiment is a set of points of a graph (Figure 3). All the points but one seem to be located on a straight line.

The point for the wire with the clamp differs from this. The amount of the difference gives the weight of the clamp.

Using functions for understanding and solving problems from practical arithmetic can be seen as a result of the effort to develop "functional thinking" more generally in mathematics education. This pedagogical idea has a long tradition in Germany (VOLLRATH 1989). Under the influence of FELIX KLEIN it had become a key concept in the resolution of the Merano conference in 1905. And the idea of function was still one of the leading concepts during the "new math" period. From the point of view of set theory, functions can be regarded as a special kind of relation.

Figure 3: graph of the length - weight function



Functions could therefore be defined in terms of relations in mathematics instruction. It seems that for a certain period didacticians and teachers were convinced that the concept of function could be learnt through such a definition. FREUDENTHAL (1973) made it clear that this is impossible. The ideas of VAN HIELE, BRUNER, and FREUDENTHAL about learning in stages of understanding became rather popular and led to new teaching strategies for concepts (VOLLRATH 1984).

Finally, the whole domain of practical arithmetic was renewed. The aspects of "mathematization" (FREUDENTHAL 1968) and "application" (ENGEL 1968) came into prominence and overcame the traditional division into discrete problem areas (e.g. percent and interest, business, commerce, succession, mixture and alloy) and the traditional dominance of methods for solving problems (e.g. rule of three). The modern view of practical arithmetic was suggested for mathematics education with the conviction that success would be guaranteed success. Therefore most of the textbooks tried to include these ideas in the 1970s. But during the last ten years didacticians have been given many hints that this was an illusion.

4. Evaluation of methods

Criticism of textbooks by teachers made it obvious that they had difficulty accepting the modern methods. They thought the function tables were too formal. As a compromise, many of them started with the rule of three and ended with the tables. And in response, most textbooks offered compromises in this direction.

Analyses of communication (ANDELFINGER 1985) made clear that there is a great difference between teacher concepts and learner concepts related to practical arithmetic. For students, schemes were still dominant. Surprisingly, it was found that they were using their own schemes. Some of these were based on their parents' explanations, others were related to the teacher's particular method. Many students were rather confused by all the different schemes. Difficulties also seem to arise from "unfriendly" numbers such as 2 2/3 kg, 12.45 DM, 0.012 m^2, which arouse feelings of anxiety.

These observations led to analyses of difficulties (VIET 1989). It was found, for example, that proportional problems are easier to solve than antiproportional problems. Quantity-cost problems are easier to handle than distance-time problems. Most of the students always assume a proportional quantity-cost

function. Considering the effect of changes in one magnitude on changes in the other seems to be more successful than studying how one magnitude depends on the other. Some of the difficulties seem to have had their origin in the great variety of methods used in the classroom, including the methods offered by the parents to master homework. Premature schematizing of the problems also seems to have had a rather bad influence.

Although problems on percentage or interest are taught in connection with fractions as an application, in many cases these problems are still solved by methods of the rule-of-three type (MEIßNER 1982, BERGER 1989). Here too the relation between the methods offered by the book, suggested by the teacher, recommended by the parents, and used by the students is rather chaotic. So the teachers are puzzled.

Difficulties which arise from calculation may be avoided to a certain degree by the use of calculators (e.g. LOERCHER, RUEMMELE 1987). Later on, when computers are used as tools in mathematics instruction, spreadsheets are recommended for problems on percentage and interest which can lead the students to "realistic" problem solving (e.g. DAUBERT 1986). But the main problem seems to be that the students do not see any practical sense in the problems; they just handle them schematically. This is a general problem of mathematics education (Baruk 1985). Some groups of teachers have started to overcome the artificiality of problems by developing "project work". This was also stimulated by the needs of comprehensive schools, which are still under discussion in the German education system. The aim is to stimulate mathematical activity through critical analyses of social problems (MÜNZINGER 1977, KANITZ 1987). Possible topics include: money, elections, traffic, leisure time and income tax.

The concept of magnitude has also been criticized (KEITEL 1979). The danger is seen that students may misunderstand quantity-cost functions as "laws of nature", which would prevent them from studying elements of cost theory. Relevant projects could perhaps overcome these limitations.

Some teachers and school administrations tend to go back to the traditional schemes. This is supported by the rather conservative organization of chambers of commerce and industry. But many didacticians still hope that they can convince the teachers and can help them to teach the modern methods more effectively. There are some books written for teachers and teacher education students which still promote the modern understanding of practical arithmetic (STREHL 1979, GLATFELD 1983, FRICKE 1987). But critical reviews hint at the limitations of this view (e.g. KEITEL 1979, GRAUMANN 1983).

Perhaps a way out may be found by more investigations into functional thinking.

5. Research about functional thinking

To learn about functions and to be successful in using functions to solve problems requires a mental ability which can be characterized as follows:

(1) Dependences between magnitudes can be stated, postulated, produced, and reproduced.

(2) Assumptions about the dependence can be made, tested, and if necessary revised.

(3) The influence of changing one magnitude on the dependent magnitude can be assumed, observed, and stated.

This ability can be called "functional thinking". Many suggestions have been made by German educators to promote functional thinking (see VOLLRATH 1989). But the evaluation of these proposals has so far been insufficient. It is felt that one needs to know more about the effects before making further recommendations.

Knowledge about the development of functional thinking can be gained from psychological studies. From PIAGET's investigations it is known that the ability

to discover the proportionality of a function develops in children (1977). He identified stages leading to *proportional reasoning*. Several studies attempted to fill gaps in the Piagetian model (see TOURNAIRE and PULOS 1985). The stages can be interpreted as stages of functional thinking. They can be defined by abilities and by limitations,such as: The child knows that from an enlargement of x an enlargement of y will result. But the child is not able to discover that a doubling of x leads to a doubling of y.

These abilities and deficiencies become apparent in problem situations in which the student is asked to predict or to precalculate a result. When a missing value problem is presented in a physical experiment with an underlying quadratic function, many children assume proportionality, for example, and have difficulty overcoming the assumption (SUAREZ 1977). To be successful in such problems it is therefore important to discover properties "beyond proportionality".

The influence of teaching methods on the ability to form assumptions about functions used in physics instruction was investigated by HÄUSSLER (1981). Experienced students typically start with the assumption of proportionality, but they also have a repertoire of further properties of functions which they can use for testing assumptions. Research on functional thinking should be aimed at yielding more information about the development of such a repertoire under the influence of mathematics instruction.

The monotonic property of an underlying function is often crucial, especially for approximate solution of a missing value problem. The development of the ability to use the monotonic property for approximations has been studied through search strategies (VOLLRATH 1986). It was found that stages of development could be identified which fit into PIAGET's scheme of stages.

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