53. Representing, Solving, and Using Algebraic Equations

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Students of beginning algebra are quickly expected to solve linear equations. The solution procedures are generally abstract, involving the manipulation of numbers and algebraic symbols. Many students, even after completing a year of algebra, do not understand variables, equations, and solving equations (cf. CARPENTER et al. 1982). One way to help students learn to solve equations is to use physical objects, diagrams, and then symbols to represent equations. (BRUNER 1964, 1967 calls such representations enactive (concrete), iconic (pictorial), and symbolic.) Although solving equations symbolically is essential, many students can benefit from working with physical problems that can also be symbolized mathematically. This article describes one way for students to learn to solve certain linear equations using pan balances, diagrams, and then symbols.

1. Pan Balances

One needs a two-pan balance, some small objects with equal weight, and small, opaque containers. The containers need to open and close. Plastic film containers and washers for the weights work especially well. The washers should fit into the containers, and one washer should weigh the same as one container. (The weight of the containers can be adjusted by drilling small holes in the bottom or gluig small washers inside the cap.) In figure 1, the lefthand pan has three containers and five washers, whereas the right-hand pan has eleven washers. Each container has the same number of washers. The goal is to determine the number of washers in each container.

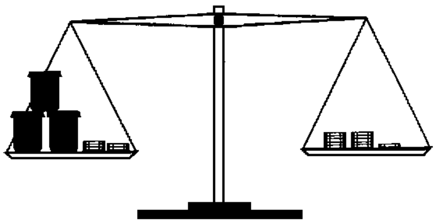

Fig. 1. A two-pan balance

One way to reach the goal is to set a single container in one pan and place enough washers in the other pan

to balance the scales. The number of washers in the container equals the number of washers in the other pan. (The container weighs one washer. When counting washers in a container, count the washers and the top of the container.)

Students are to manipulate the items in the pans so that after each manipulation, or transformation, the pans still balance. By trial and error, students quickly see that it is possible to add or remove objects of equal weight from each side. Using w to represent the number of washers in each container, the arrangement in figure 1 can be represented algebraically as

$$3w + 5 = 11.$$

One way to proceed is to remove five washers from each balance pan; the two sides still balance. Algebraically,

$$3w = 6.$$

Although we can remove two containers and the required number of washers to maintain a balance, it is more useful to have students tell how many items to remove before beginning. As three containers are involved, divide the six washers into three equal piles, each with two washers. Remove two of the three piles and two of the three containers, so

$$w = 2.$$

Open the container to "check" the answer. (Remember, count the lid as one washer.)

In an introductory algebra class, students used a balance to do the following tasks in the order listed:

- Represent equations and variables
- Set up the balance to represent equations and "check" solutions, that is, determine if the pans balance
- Solve equations
- Algebraically solve equations, sketching the balance pans with containers and washers to represent each step of the solution

When "checking" whether 2 is a solution to

$$2(x + 1) = 6,$$

one student could not get the pans to balance. The student had represented the left side with two containers and only one washer. A discussion of the distributive property followed. When solving equations using the balance, teams were used. One team set up the balance for another team to solve.

Balances can be used to illustrate the "uniqueness" of a solution. Ask, if one washer is added to or removed from each container in figure 1, will the pans still balance? Then do the experiment to see the result. Next set up the balance corresponding to the equation

(1) $$2(w+1)=2w+2$$

and repeat the experiment. This approach demonstrates that some equations have many solutions.

Introductory algebra students had considerable difficulty solving equation (1). They had expected that all equations could be simplified to an equation with the variable equal to a number. When the procedure resulted in two empty pans, the students were certain that they had made an error. Only after putting different numbers of washers in the containers in the original representation of (1) and checking that the pans still balanced did students understand that it was not possible to solve and get only one solution. Writing the simplified equation as $0w = 0$ rather than $0 = 0$ also helped students understand the solution.

Problems with containers of different colors can be given. For example, use r to represent the number of washers in the red containers. Whereas students readily accept that the color of the containers does not affect the solution, some students think that the letter used to represent a variable affects its value (cf. WAGNER 1981). For example, if x is an integer, some students believe that y is the next consecutive integer!

With more advanced students, systems of linear equations can be represented with two balances. Using red and green containers, figure 2 represents the system of equations

(2) $$\begin{cases} g+r = 7, \\ 2g+1 = r. \end{cases}$$

Here g is the number of washers in each green container and r the number in each red container. (Count the lid as a washer here also.) To solve the problem using balances, students must use substitution.

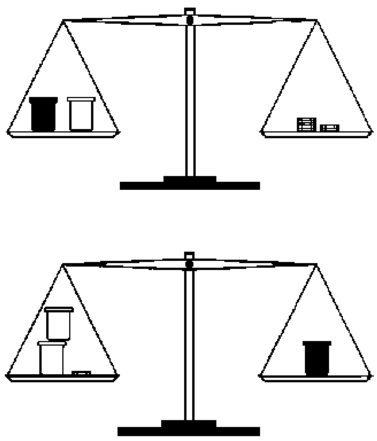

Fig. 2. A presentation of two-variable, two-pan-balance problem

The balance uses physical objects to represent equations and to lead students to algebraic solutions of equations. The balance also serves as a check for each step in a solution, since the pans must balance each time. A disadvantage of the balance is that solutions are always positive; empty containers weigh one washer. Also equations involving subtraction are not easily modeled; springs or balloons can be used to pull up a side of the balance but may not be worth the added complications. However, with the balance students can conceptualize the steps in a solution physically as well as abstractly.

2. Pictoral Representations

Following an idea of Papy, an arrow diagram can be used to represent and solve many equations. The procedure does not seem to be used in American texts but is widely used in introductory German algebra (cf. VOLLRATH 1980). VOLLRATH (1986, 124) uses the procedure in a German seventh grade mathematics text. Figure 3 shows the representation of

$$3w + 5 = 11.$$

Beginning with the variable, the left-hand side of the equation is developed one step at a time.

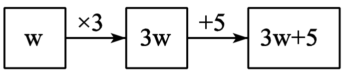

Fig. 3. Expanding $3w + 5$

The right-hand side of the equation, 11, is written under the final expression, 3w + 5.

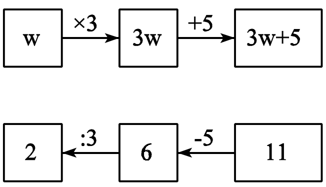

Fig. 4. Solving $3w + 5 = 11$

The solution is found by working back-ward with inverse operations. See figure 4.

The following equations, or usual solution, are obtained by placing equal signs between the corresponding boxes:

$$3w + 5 = 11$$

$$3w = 6$$

$$w = 2$$

This pictorial representation shows students the relation between the order of

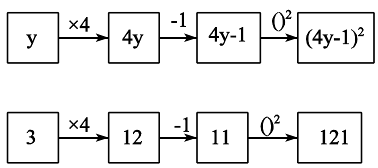

Fig. 5 Evaluating $(4y - 1)^2$ for $y = 3$.

operations and the solution of equations.

The representation can also be used to evaluate expressions. Figure 5 illustrates evaluating $(4y - 1)^2$ when $y = 3$.

When the variable occurs more than once in a linear equation, a variation of

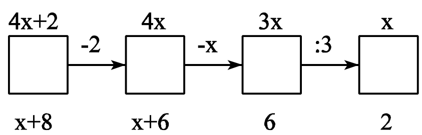

Fig. 6. Solving $4x + 2 = x + 8$

this procedure can be used. Figure 6 shows a solution of the equation

$$4x + 2 = x + 8.$$

Another pictorial representation uses line segments. The representation links equations to the usual method of introducing addition and subtraction of positive and negative numbers and links addition with subtraction as well as

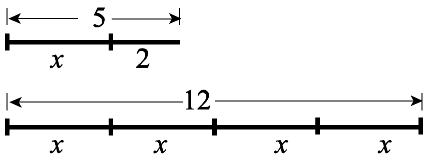

Fig. 7. Representing equations using line segments

multiplication with division. It also links algebraic and geometric representations of equations. Figure 7 illustrates this representation. The diagrams in figure 7 can be interpreted as representing several equations. Possible equations for the first diagram are

$$x + 2 = 5,$$

$$x = 5 - 2,$$

and

$$5 - x = 2.$$

Possible equations for the second diagram are

$$4x = 12,$$

$$x = \frac{12}{4},$$

and

$$\frac{12}{x} = 4$$

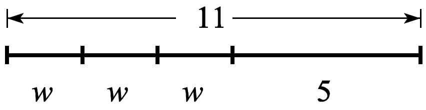

Fig. 8. Representing
$3w + 5 = 11$ and $3w = 11 - 5$
using line segments

These line-segment representations and equations give another procedure for solving equations. Figure 8 again represents the equation

$$3w + 5 = 11.$$

Figure 8 is also a representation of

$$3w = 11 – 5,$$

which gives

$$3w = 6.$$

Figure 9 represents the foregoing equation and also the equation

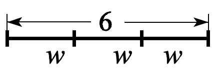

Fig. 9. Representing
$3w = 6$ and $w = 6/3$
using line segments

$$w = \frac{6}{3},$$

or

$$w = 2.$$

The solution can be checked in the original equation. Try using this method to solve

$$13 - 3y = 7.$$

3. Abstract or Symbolic Representation

Often the only way equations are represented and solved in algebra is abstractly or symbolically. However, using only a purely symbolic representation and solution makes it hard for some students to understand equations and why each transformation is used in solutions.

4. Applications of Equations

In problem solving, students must understand not only how to solve equations but also how to use equations. Following VOLLRATH (1984, 123), equations, as well as many other concepts, have multiple uses. Teachers should pomt out the various uses so that students understand how fundamental equations are to problem solving. The uses of equations include the following:

1. A source of a problem

   Example. Solve the equation $3x + 2 = 14$.

2. An aid in solving a problem

   Example. Three more than twice a number is 15. Find the number.

   Solution. Let n be the number. Translating the problem to an equation gives

   $2n + 3 = 15$. Solving gives that n is 6.

3. A solution to a problem

   Example. Find the set ofpoints (x, y) equidistant from the points (3, 2) and (0, 1).

   Solution. The distance formula gives each (x, y) that satisfies the equation $y = -3x + 6$.

4. A check to determine if a solution exists or is unique

   Example. Find the solution(s) to these equations:

   $$3x + 2y = 23$$

   $$5x - 2y = 17$$

   Solution. As the equation from the determinant

   $$\begin{vmatrix} 3 & 2 \\ 5 & {}^{-}2 \end{vmatrix} = {}^{-}6 - 10 = {}^{-}16$$

   is not zero, a unique solution exists. Solving using any procedure studied gives

   $$x = 5 \text{ and } y = 4.$$

5. A check on an estimated or possible solution

   Example. Which function satisfies $f(ax) = af(x)$?

(i) $f(x) = x + 2$

(ii) $f(x) = 2x$

(iii) $f(x) = 2$

Solution. Substitute each possible solution into the equation and determine if an identity results.

5. **Summary**

Variables and equations are key concepts in algebra and in mathematics. Students need to understand the procedures for solving equations and how equations are used. Students can benefit from being shown the many uses of equations in problem solving. We suggest the use of concrete and pictorial representations, as well as the symbolic representation of equations, with introductory classes. Such an approach might result in students´ better understanding the steps in solving equations. Some older students might understand for the first time why equations are solved in the way they have long practiced.

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